Polynomial Division in Number Theory: Hints

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Canadian Winter Camp 2017

1. (PEN A6, CRUX no.1746) Do cases $a = b$, $a = 1$, and now assume $1 < a < b$. Write $\frac{a^2+b^2-1}{ab} = \frac{b}{a} + \frac{a^2-1}{ab}$.

2. (PEN A109, APMO 2002) WLOG $a \geq b$, and then note that $0 < \frac{b^2+a}{a^2-b} \leq \frac{a^2+a}{a^2-a} = \frac{a+1}{a-1}$. Divide into cases $a = 2$, $a = 3$, and $a \geq 4$.

3. (BMO1 2016) We have $n \mid m$, so write $m = an$ for some $a \in \mathbb{Z}^+$. Now $n + 2 \mid an + 1$ so $n + 2 \mid 2a - 1$, so $2a - 1 = b(n + 2)$ for some odd positive integer b. Repeat in this fashion.

4. (IMO 1988 variant) Rewrite as $\frac{x^2+y^2}{xy-1} = N$ for x, y, N positive integers, where we need to show $N = 5$. WLOG $x \geq y$, and write $N = \frac{x}{y} + \frac{y^3+x}{y(xy-1)}$, and show that the nasty term is in $(0, 2)$. Write $x = ny - r$ for $n \geq 2$, $0 \leq r < y$ and use this.

5. (PEN H9, Ireland 1995) WLOG $x, y > 0$, where we also must negate any a's found (as (x, y, a) works if any of $(-x, -y, a)$,$(x, -y, -a)$,$(-x, y, a)$ does. Now WLOG $x > y$ and then $a = \frac{1-x^2-y^2}{xy} = -\frac{x}{y} + \frac{1-y^2}{xy}$ give the nasty and nice terms.

6. (PEN A91, IMO 1998) Write $\frac{a^2b+a+b}{ab^2+b+7} = \frac{a}{b} + \frac{b^2-7a}{ab^3+b^2+7b}$, do cases $b = 1$, $b > 1$ with subcases on whether $b^2 - 7a$ is nonnegative or negative.

7. (PEN A77, Russia 2001) Solve case $x \leq y$ by showing $\frac{x^2+y}{xy+1} < 2$. For $y < x$, write $\frac{x^2+y}{xy+1} = \frac{x}{y} + \frac{y^2-x}{y(xy+1)}$.

8. (PEN A5) WLOG $x \geq y$, and write $\frac{x^2+y^2+1}{xy} = \frac{x}{y} + \frac{y^2+1}{xy}$. Get another descent argument.

9. (PEN A89, Turkey 1994) WLOG $a \geq b$, do cases $b = 1, 2, 3$ and $a = b$. Then write $\frac{a^2+b^1+3}{ab} = \frac{a}{b} + \frac{b^2+3}{ab}$.

10. (PEN A78, IMO 1994) Show (m, n) works if and only if (n, m) does. Then assume $m \geq n$, and write $\frac{n^3+1}{mn-1} = \frac{n^2}{m} + \frac{n^2+m}{m(mn-1)}$.

11. (PEN A79, IMO 2003) Divide into cases $2a - b = 0$, and $2a - b > 0$. Do $b = 0$, then show $|a| > |b|$. Show $a > 0$ so that $a > |b| > 0$. Then divide out as $Q = \frac{a^2}{2ab^2-b^3+1} = \frac{a}{2b^2} + \frac{\frac{b^3-1}{2b^2}a}{b^2(2a-b)+1}$, show that the nasty term is in $(-1, 1)$. Write $a = 2b^2n + r$ with $0 \leq r < 2b^2$. Replacing a, we see that we get

$$Q = n + \frac{n^3b + 2rnb^2 + r^2 - n}{4nb^4 + 2rb^2 - b^3 + 1},$$

where this second term must be 1 is $b \geq 1$, and 0 if $b \leq -1$. If $b \leq -1$ show $(a, b) \to (-r, b)$ and use that $-r \leq 0$ to deduce (a, b). If $b \geq 1$, show $(a, b) \to (2b^2 - r, b) = (u, b)$. Deduce solutions from $2u - b = 0$, and otherwise show that $0 < Q < 1$ is true if $3b^3 > 4b^2$ (manipulate and use $2u > b$). So we only get more solutions from $b = 1$.

12. (IMO 2015) WLOG do $a \leq b \leq c$, write $ab - c = 2^x$, $ac - b = 2^y$, $bc - a = 2^z$, so $x \leq y \leq z$. The key point now is that $ab - c \mid ac - b$, and $ac - b \mid bc - a$. Consider taking $c = ab - 2^x$, and putting that into $2^{z-y} = \frac{bc-a}{ac-b}$, and take out a "nice" term of $\frac{ab}{a^2-1}$. With a bit of algebra you can show that $(a^2-1)2^{z-y} = ab+\epsilon$, where $\epsilon \in \{-1, 0, 1\}$ (you will need a number theoretic argument to show that $a^2 \geq 2^x + 1$). $\epsilon = 1$ should

lead to $(3,5,7)$, $\epsilon = -1$ leads to $(2,2,2),(2,2,3)$, and $\epsilon = 0$ gives $(2,6,11)$. For some cases you will need to consider when sums of powers of 2 are equal, i.e. binary.

12'. (IMO 2015 variant) The proof of 12 should pretty much carry over, except in the casework you get no solutions as opposed to the 4 you got when $p = 2$.

12". (IMO 2015 variant) Apply similar arguments to the above.